COMBINATORIAL ANALYSIS SITUS*

BY

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1. **Introduction.** This is the first part of a paper on the topological theory of complexes. We begin by setting up a criterion for the homeomorphism of two complexes, expressed in terms of so-called *elementary transformations*, which are combinatorial in character. We then give a short account of the classical theory of connectivity and of a modified form of the theory where we operate with chains reduced modulo π. The theory of connectivity, modulo π, leads to connectivity numbers, modulo π, which are equivalent to the combined connectivity numbers and coefficients of torsion of the older theory. We close with strictly combinatorial proofs of the invariance of the connectivity numbers, modulo π.

The second part of the paper will be devoted to the theory of intersecting chains, out of which theory new topological invariants will be derived that cannot be expressed in terms of the classical ones.† Here, the advantage of operating with chains, modulo π, will become more apparent.

I. Combinatorial formulation

2. **Simplex and complex.** A *k-simplex* is the k-dimensional analogue of a tetrahedral region. Thus, for example, a 0-*simplex* is a point, a 1-*simplex* a line segment, a 2-*simplex* a triangular plane region, and so on. The *boundary* of a k-simplex is made up of simplexes of dimensionalities $0, 1, \cdots, k-1$. It comprises $k+1$ 0-*faces* (*vertices*), $\frac{1}{2}(k+1)k$ 1-*faces* (*edges*), $\cdots$, $(k+1)!/(i+1)!(k-i)!$ i-*faces*, $\cdots$. For reasons of symmetry, we shall say that a k-simplex is its own k-face, though we shall not regard it as a member of its own boundary. A k-simplex is completely determined by its vertices. If we denote the latter by the marks $V_0, V_1, \cdots, V_k$ respectively, we may conveniently denote the simplex itself by the symbol

$$|V_0V_1 \cdots V_k|, \tag{2.1}$$

where the order of appearance of the marks V_i is immaterial.

For the purposes of this paper, a *complex* will be any finite set of simplexes such that (1) no two simplexes of the set have a point in common,

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† Cf. abstracts in the Proceedings of the National Academy of Sciences, vol. 10 (1924), p. 99, p. 101, p. 493; vol. 11 (1925), p. 143.

(2) every face of a simplex of the set is itself a simplex of the set. This definition is less inclusive than the one ordinarily given.* However, there is no difficulty in showing that every complex of the more general sort is homeomorphic† with one of the restricted type defined above, so that, topologically speaking, no essential loss of generality results from thus narrowing down the definition.

The k-simplexes of a complex ($k=0, 1, 2, \cdots$) will be called the *cells* of the complex. 0-cells and 1-cells will also be called *vertices* and *edges* respectively. A complex is completely determined by those of its cells that are not on the boundaries of other cells; the presence of the remaining cells follows automatically from the definition of a complex. If the determining cells of a complex are all of the same dimensionality n, that is to say, if the complex consists entirely of n-cells and the cells on their boundaries, we shall call it an n-complex.

A complex may be represented schematically by the combined symbols of its determining cells and, therefore, ultimately, in terms of the symbols of its vertices. *We shall so arrange that all operations on complexes are ultimately expressible as combinatorial operations on the symbols associated with the vertices of the given complexes and of complexes derived from the latter by subdivision.*

A complex will be said to be *connected* provided that if V and V' are any two of its vertices there is always a sequence of vertices

$$V, V_1, V_2, \cdots V_i, V'$$

beginning with V and ending with V' such that every pair of consecutive vertices of the sequence is the boundary of an edge of the complex.

3. **Derived complexes.** Let

$$|V_0V_1 \cdots V_i| \qquad (i>0), \tag{3.1}$$

be an i-face of a k-simplex

$$|V_0V_1 \cdots V_iV_{i+1} \cdots V_k| \qquad (k \geqq i), \tag{3.2}$$

and let W be the center of mass of the face (3.1). We may then subdivide the simplex (3.2) into the $(i+1)$ k-simplexes

$$\begin{array}{l} |\, WV_1 \cdots V_iV_{i+1} \cdots V_k \,| \\ |\, V_0W \cdots V_iV_{i+1} \cdots V_k \,| \\ \cdots\cdots\cdots\cdots\cdots \\ |\, V_0V_1 \cdots WV_{i+1} \cdots V_k \,| \end{array} \tag{3.3}$$

* Cf., for example, Veblen, *Cambridge Colloquium Lectures*, Part I, *Analysis Situs*, p. 72. The symbol *VC* will hereafter be used to denote a reference to these Lectures.

† §4. *VC*, pp. 85, et seq.

together with certain separating faces of lower dimensionalities which need not be indicated explicitly. Furthermore, if the simplex (3.1) is a cell of a complex Φ, we may perform a subdivision of the type just described upon each cell (3.2) of the complex having the cell (3.1) as a face, whereby we shall transform the complex Φ into a new complex Ψ, made up of cells and sub-cells of Φ. The operation transforming the complex Φ into the complex Ψ will be called an *elementary subdivision*. It will be noticed that an elementary subdivision may be represented schematically in terms of vertex symbols, in conformity with the general plan of procedure laid out in § 2. For reasons of symmetry, we shall do away with the restrictions that the simplex (3.1) be of greater dimensionality than zero. When (3.1) is a 0-simplex, the operation of elementary subdivision becomes trivial and amounts merely to changing the name of the vertex V_0 from V_0 to W.

Now, suppose we assign an order to the cells of a complex Φ such that no cell precedes another of higher dimensionality than its own. Suppose, moreover, that we perform a series of elementary subdivisions, one corresponding to each cell of Φ, in such a manner that the new vertex W_i introduced in making the ith subdivision lies, in each case, at the center of mass of the ith cell of Φ (with respect to the ordering assigned to the cells of Φ). The effect of these combined operations is to transform the complex Φ into a new complex Φ_1, made up of proper sub-cells of Φ and having one vertex at the center of mass of each cell of Φ. We shall call the complex Φ_1 the *first derived complex* of Φ. The complex Φ_1 is independent of the exact ordering assigned to the cells of Φ so long as the condition is fulfilled that no cell shall be preceded by another of lower dimensionality than its own. The *nth derived complex* Φ_n of a complex Φ will be defined, by induction, as the first derived complex of the $(n-1)$st derived complex Φ_{n-1} of Φ. The derived complexes of Φ form an infinite sequence

$$\Phi_1 , \Phi_2 , \Phi_3 , \cdots$$

which will be called the *derived sequence* of Φ. We notice that the maximum diameter of the cells of Φ_i approaches zero as i increases without bound.

The vertices of the derived complexes of Φ will be called the *derived vertices* of Φ. They and their limit points make up the points of Φ.

4. **Homeomorphism in terms of derived vertices.** Two complexes are said to be *homeomorphic* provided there exists a one-one continuous correspondence between the points of one and the points of the other.

A necessary and sufficient condition that two compexes Φ and Ψ be homeomorphic is that there exist a one-one continuous correspondence between the derived vertices of Φ and of Ψ.

The sufficiency of the condition is obvious, because a one-one continuous correspondence between the derived vertices of Φ and Ψ induces a similar correspondence, by continuity, between all the points of Φ and of Ψ. To prove that the condition is necessary, we shall assume that there exists a one-one continuous correspondence K between the points of Φ and of Ψ and prove, in consequence, that there exists a similar correspondence between the derived vertices.

Let Φ_i be any derived complex of Φ. We shall first prove that there exists a correspondence K_1 differing from the given correspondence K by as little as we please and such that all the vertices of the complex Φ_i are mapped by K_1 upon vertices of some derived complex Ψ_j of Ψ. Consider the set of derived complexes Ψ_j of Ψ. Since the maximum diameter of the cells of a derived complex Ψ_j approaches zero as j increases without bound, it is possible to choose the index j in such a manner that the images with respect to K of no two vertices of the complex Φ_i lie on the same cell of the derived complex Ψ_{j-1} immediately preceding Ψ_j. Now, if we wish to form the complex Ψ_j from the complex Ψ_{j-1}, we must introduce a new vertex W_s at the center of each cell of Ψ_{j-1}. Instead of doing this, however, let us agree, whenever a cell of Ψ_{j-1} contains the image P_t of a vertex of Φ_i, to introduce the new vertex W_s at the point P_t rather than at the center of the cell in question. Then if we proceed in other respects, just as we would if we were forming the complex Ψ_j, we are led to a complex Ψ'_j, similar to Ψ_j in cellular structure and deformable into Ψ_j by merely shifting the vertices W_s back to the centers of the respective cells of Ψ_{j-1} on which they lie. The deformation D undergone by the complex Ψ when Ψ'_{j-1} is transformed into Ψ_j clearly approaches zero as j increases without bound, since no point of Ψ ever leaves the cell of Ψ_{j-1} on which it lies initially. Since the correspondence K maps the vertices of the complex Φ_i upon vertices of the complex Ψ'_{j-1}, the correspondence $K_1 = KD$ maps the vertices of the complex Φ_i upon vertices of the complex Ψ_j and is, therefore, the correspondence we set out to find.

By an immediate induction, we now see that there exist an infinite sequence of complexes

$$\Phi_{i_1}, \Psi_{i_2}, \Phi_{i_3}, \Psi_{i_4}, \cdots \tag{4.1}$$

and an infinite sequence of one-one continuous correspondences

$$K_1, K_2, K_3, K_4, \cdots \tag{4.2}$$

such that

(i) the odd and even terms of (4.1) are sub-sequences of the derived sequences Φ and Ψ respectively;

(ii) the ith correspondence K_i maps the vertices of the $(i+1)$st complex of (4.1) upon vertices of the ith;

(iii) if the correspondence K_i pairs a derived vertex U of Φ with a derived vertex V of Ψ, all subsequent correspondences K_{i+s} pair the vertices U and V;

(iv) the sequence (4.2) converges uniformly to a limiting correspondence K_∞.

The limiting correspondence K_∞ is necessarily one-one continuous, by a well known theorem on uniformly convergent sequences. Moreover, it pairs the derived vertices of Φ with the derived vertices of Ψ in the manner required by the theorem.

5. **Elementary transformations.** If V is a vertex of a complex Φ we define the *star* $S(V)$ of Φ with center at V as the set of all cells of Φ having V as a vertex. Evidently, a 0-cell belongs to exactly one star of Φ, a 1-cell to two, a 2-cell to three, and so on. The *boundary* of a star will be the complex composed of the cells of Φ that are on the boundaries of cells of the star without themselves belonging to the star. Each i-cell of a star $(i>0)$ has exactly one $(i-1)$-face on the boundary of the star, and each $(i-1)$-cell on the boundary of the star is the $(i-1)$-face of exactly one i-cell of the star.

A set of vertices will be said to be *mutually adjacent* if, and only if, they are the vertices of a cell of Φ. It should be observed that a set of vertices may fail, as a whole, to be mutually adjacent in spite of the mutual adjacency of every pair of vertices in the set. Thus, for example, the three vertices of a triangle are not mutually adjacent with respect to the complex made up of the vertices and sides of the triangle, though they become mutually adjacent if the plane triangular region bounded by the triangle is added to the complex.

A necessary and sufficient condition that a set of vertices be mutually adjacent is that they be the centers of stars having at least one cell C^i in common. For if all their stars contain the cell C^i, they must themselves be vertices of C^i and, therefore, the vertices of some face of C^i. Conversely, if the vertices of the set determine a face of C^i, the cell C^i obviously belongs to all of their stars, from the very definition of a star. This simple condition for mutual adjacency will be useful in the sequel.

Now, consider any single-valued transformation τ carrying the vertices of a complex Φ into vertices of a complex Ψ in such a manner that sets of mutually adjacent vertices of Φ are carried into sets of mutually adjacent vertices of Ψ. From the definition of mutual adjacency, it follows that the vertices $V_0, V_1, \cdots, V_i$ of a cell of Φ are carried by τ into the vertices

$W_0, W_1, \cdots, W_i$ of a cell of Ψ. We may, therefore, extend the domain of definition of the transformation τ to the entire complex Φ by prescribing that it shall carry the interior and boundary of each cell

$$|V_0V_1 \cdots V_i| \tag{5.1}$$

of Φ by an affine transformation into the interior and boundary of the corresponding cell

$$|W_0W_1 \cdots W_i| \tag{5.2}$$

of Ψ, where the transformation is determined by the condition that V_0 be carried into W_0, V_1 into W_1, and so on. The extended transformation τ is single-valued and continuous over Φ; we shall call it an *elementary transformation.*

In defining an elementary transformation, *we do not wish to imply that distinct vertices of* Φ *are transformed into distinct vertices of* Ψ. In other words, it may very well happen that some of the vertices W_i in (5.2) coincide, and that the i-cell (5.1) of Φ is carried by a *degenerate* affine transformation (with vanishing determinant) into what we may call a *degenerate* i-cell (5.2) of Ψ,— properly speaking, a cell of dimensionality less than i. Thus, the inverse of the transformation τ need neither be single-valued nor everywhere defined over the complex Ψ. An elementary transformation is completely determined by its action on the vertices of a complex; consequently it is representable by a combinatorial operation on vertex symbols, again in accordance with our general program.

6. **Approximation of a continuous transformation.** Now, consider an arbitrary single-valued continuous transformation T of a complex Φ into a complex Ψ. (The inverse of the transformation T is not assumed to be single-valued or everywhere defined on Ψ.) We shall prove that there exists an infinite sequence of elementary transformations

$$\tau_1, \tau_2, \tau_3, \cdots \tag{6.1}$$

converging uniformly to the transformation T.

Let

$$\Phi_1, \Phi_2, \Phi_3, \cdots \tag{6.2}$$

and

$$\Psi_1, \Psi_2, \Psi_3, \cdots \tag{6.3}$$

be the derived sequences of Φ and Ψ respectively (§3). Then, if Ψ_i is any complex of the second sequence, there exists a complex Φ_j of the first such that the image with respect to the transformation T of each star of Φ_j

is completely covered by a star of Ψ_i. For the maximum diameter of a cell, and therefore of a star, of Φ_j approaches zero as j increases without bound; hence, the same is true of the maximum diameter of the image of a star of Φ_j, by elementary principles of uniform continuity. Thus, we see that there exists an infinite sub-sequence

$$\Phi_1' , \Phi_2' , \Phi_3' , \cdots \qquad (\Phi_i' = \Phi_j) \tag{6.4}$$

of the sequence (6.2) such that, for every value of i, the image with respect to T of a star of Φ_i' is covered by a star of Ψ_i.

The transformation τ_i in (6.1) may now be determined by the condition that it shall carry the vertices of Φ_i' into vertices of Ψ_i in such a manner that the center of each star S^Φ of Φ_i' is transformed into the center of one of the stars S^Ψ of Ψ_i covering the image of the star S^Φ with respect to the transformation T. To prove that this condition actually determines an elementary transformation, it is sufficient to show that mutually adjacent vertices of Φ_i' are carried into mutually adjacent vertices of Ψ_i. This follows at once, however, from the test for mutual adjacency given in § 5. For if a set of stars S_t^Φ of Φ_i $(t=1, 2, \cdots)$ have a cell C in common, so also must the corresponding stars S_t^Ψ of Ψ_i covering the images of the stars S_t^Φ respectively, since the stars S_t^Φ all cover the image of the cell C. Finally, the transformations τ_i converge uniformly to the transformation T, because the image of a point P of Φ with respect to τ_i differs from the image of P with respect to T by less than the maximum diameter of a star of Ψ_i. But this maximum diameter approaches zero as i increases without bound. This completes the argument.

If the complexes Φ and Ψ are identical, the transformation T is a transformation of the complex Φ into itself. Now, it may happen that the transformation T is the identity. In this case, the approximating transformation τ_i carries each vertex V of the complex Φ_i' into the center of a star of the complex $\Phi_i(\Phi_i = \Psi_i)$, on which the vertex V lies, that is to say, into a vertex of the cell of Φ_i on which the vertex V lies. Such a transformation will be said to be *pseudo-identical*. We notice that it leaves fixed all the vertices of the derived complex Φ_i of Φ. Clearly, an infinite sequence of pseudo-identical transformations

$$\tau_{i_1} , \tau_{i_2} , \tau_{i_3} , \cdots \qquad (i_n > i_{n-1}) ,$$

converges uniformly to the identity, since the transformation τ_{i_n} displaces no point of the complex Φ by more than the maximum diameter of a cell of Φ_{i_n}.

7. **A condition for homeomorphism.** We are now ready to state in terms of elementary transformations a necessary and sufficient condition that two complexes Φ and Ψ be homeomorphic. To derive a necessary condition, let us assume that there exists a one-one continuous correspondence K between the points of Φ and the points of Ψ. Let

$$\Phi_1 , \Phi_2 , \Phi_3 , \cdots \tag{7.1}$$

and

$$\Psi_1 , \Psi_2 , \Psi_3 , \cdots \tag{7.2}$$

be the derived sequences of Φ and Ψ respectively. Then, if $\Delta_1 = \Phi_i$ is any complex of the first sequence, there exists a complex $\Delta_2 = \Psi_j$ of the second such that the image with respect to K of each star of Δ_2 is covered by a star of Δ_1 (cf. § 6). Hence, by induction, there exists an infinite sequence of complexes

$$\Delta_1 , \Delta_2 , \Delta_3 , \cdots \tag{7.3}$$

with terms chosen alternately from (7.1) and (7.2), such that the image with respect to K of each star of each complex $\Delta_{i+1} (i>0)$ is covered by a star of the immediately preceding complex Δ_i. By the process of § 6, we may, therefore, determine an infinite sequence of elementary transformations

$$\tau_1 , \tau_2 , \tau_3 , \cdots , \tag{7.4}$$

where the transformation τ_i carries the vertices of Δ_{i+1} into vertices of Δ_i and has the property that if a vertex V_{i+1} of Δ_{i+1} is carried into a vertex V_i of Δ_i the image with respect to K of the star of Δ_{i+1} with center at V_{i+1} is covered by the star of Δ_i with center at V_i.

Now, the transformations τ_{2i+1} of odd orders map the points of the complex Ψ upon points of the complex Φ and the transformations τ_{2i} of even orders map the points of the complex Φ upon points of the complex Ψ. Therefore, each product transformation of the form $\tau_{i+1}\,\tau_i$ maps one or the other of the complexes Ψ and Φ upon itself according as i is even or odd. Moreover, the transformation $\tau_{i+1}\,\tau_i$ carries the center V_{i+2} of a star S_{i+2} of Δ_{i+2} into the center of a star S_i of Δ_i such that S_i covers S_{i+2}. Therefore, since the center of a star is a vertex of all the cells of the star, the transformation $\tau_{i+1}\,\tau_i$ carries each vertex V_{i+2} of Δ_{i+2} into a vertex of the cell of Δ_i on which the vertex V_{i+2} lies. In other words, the transformation $\tau_{i+1}\,\tau_i$ is pseudo-identical.

The existence of the two sequences (7.3) *and* (7.4) *such that all product transformations of the form* $\tau_{i+1}\,\tau_i$ *are pseudo-identical is both a necessary and a sufficient condition that the complexes* Φ *and* Ψ *be homeomorphic.*

We have just seen that the condition is necessary; let us next show that it is sufficient. We prove, first of all, that the sequence

$$\tau_1 , \tau_3 , \tau_5 , \cdots \tag{7.5}$$

consisting of the odd terms of (7.4) converges uniformly to a definite limiting transformation T of the points of Ψ into points of Φ. This is done by comparing two transformations τ_i and τ_j of (7.5) (i, j odd, $j>i$) with the auxiliary transformation

$$\tau_{ji}=\tau_j\tau_{j-1}\cdots\tau_{i+1}\tau_i .$$

This last transformation may be written in the form

$$\tau_{ji}=\tau_j(\tau_{j-1}\cdots\tau_i) ,$$

where the expression in the parentheses is a pseudo-identical transformation of Φ carrying a point of a cell of Δ_i into a point on the boundary of the same cell. Therefore, the image of a point Q of Ψ with respect to τ_j differs from the image of the same point with respect to τ_{ji} by no more than the maximum diameter of a cell of Δ_i. But the transformation τ_{ji} may also be written

$$\tau_{ji}=(\tau_j\cdots\tau_{i+1})\tau_i$$

where, this time, the expression in the parentheses is a pseudo-identical transformation of Ψ carrying a point Q of a cell C of Δ_{i+1} into a point, or boundary point, of the same cell C. Hence, since the transformation τ_i carries the points and boundary points of the cell C of Δ_{i+1} into points or boundary points of a single cell of Δ_i, the image of Q with respect to τ_i also differs from the image of Q with respect to τ_{ji} by no more than the maximum diameter of a cell of Δ_i. By combining these two results, we see that the image of Q with respect to τ_i differs from the image of Q with respect to τ_j by no more than twice the maximum diameter of a cell of Δ_i, regardless of how j may be chosen. But the maximum diameter of a cell of Δ_i approaches zero as i increases without bound. Therefore, the uniform convergence of the sequence (7.5) is established. By a standard theorem on uniformly convergent sequences of continuous transformations, the limiting transformation T determined by (7.5) is also a continuous transformation. In a similar manner, we show that the sequence

$$\tau_2 , \tau_4 , \tau_6 , \cdots \tag{7.6}$$

composed of the even terms of (7.4) converges uniformly to a continuous transformation U of the points of Φ into points of Ψ.

Now, the transformations T and U are, of course, single-valued, as well as continuous, over the complexes Ψ and Φ respectively. We know, more-

over, that $\tau_{i+1}\tau_i$ (for odd values of i) converges uniformly as i increases indefinitely, both to the identity and to UT. Therefore, we have

$$T^{-1}=U\ ,$$

which proves that the inverse of T is single-valued, as well as T itself. In other words, the transformation T determines a one-one continuous correspondence between the points of Φ and the points of Ψ. Hence, finally, the complexes Φ and Ψ are homeomorphic.

The above test for homeomorphism involves a limiting process. In proving the invariance of the topological constants, all that we shall actually need will be the following simpler, and purely combinatorial, theorem.

Let Φ and Ψ be homeomorphic complexes. Then, if Φ_i is any derived complex of Φ there exist derived complexes Ψ_j and Φ_k of Ψ and Φ_i, respectively, and elementary transformations τ and τ' such that

(i) *the transformation τ maps mutually adjacent vertices of Ψ_j upon mutually adjacent vertices of Φ_i;*

(ii) *the transformation τ' maps mutually adjacent vertices of Φ_k upon mutually adjacent vertices of Ψ_j;*

(iii) *the transformation $\tau'\tau$ is pseudo-identical.*

This theorem is contained in the previous one.

II. Connectivity

8. **Chains.** An *elementary i-chain* of a complex Φ will be defined as any symbolical expression of the form

$$\pm V_0V_1\cdots V_i\ , \tag{8.1}$$

where the marks $V_0, V_1, \cdots, V_i$ denote the vertices of an i-cell of Φ. In the expressions (8.1), we take account of the order in which the marks V_i are written, but make the convention that wherever two marks are permuted, the expression as a whole merely changes in sign. Thus, for example,

$$V_0V_1V_2=-V_1V_0V_2=V_1V_2V_0=\cdots\ .$$

It follows from this convention that there are two, and only two, essentially distinct elementary i-chains (8.1) associated with each i-cell $|V_0V_1\cdots V_i|$ of Φ.

Now, there are two distinct *orientations* on an i-cell $|V_0V_1\cdots V_i|$, just as there are two distinct directions on a line segment $|V_0V_1|$. It is therefore feasible to associate the symbol $V_0V_1\cdots V_i$ with the cell $|V_0V_1\cdots V_i|$ taken with one orientation and the symbol $-V_0V_1\cdots V_i$

with the same cell taken with the other. We prefer, however, to treat the expressions $\pm V_0V_1 \cdots V_i$ as purely symbolical, so as not to go into the question of just what is meant by an oriented cell. Let us merely remark, in passing, that the notion of oriented cells plays an important rôle in the theory of integration. If an i-cell is the domain of integration of a multiple integral, the sign of the integral depends upon the orientation of the i-cell.

Now, let the elementary i-chains (8.1) of the complex Φ be denoted by the abridged symbols

$$\pm E_s^i \qquad (s=1,2, \cdots, \alpha^i),$$

respectively. Then, any linear combination

$$K^i = x^1 E_1^i + x^2 E_2^i + \cdots + x^{\alpha^i} E_{\alpha^i}^i \tag{8.2}$$

of elementary i-chains E_s^i with arbitrary integer coefficients x^s will be called an *i-chain* of Φ. We shall ordinarily write an expression such as (8.2) in the condensed form

$$K^i = x^s E_s^i, \tag{8.3}$$

where, by a convention similar to the one now current in tensor analysis, the double appearance in the same term of any variable index s, once as a subscript and once as a superscript, will be taken to imply a summation with respect to the index in question.

The i-chain K^i may be pictured as the domain of integration of an i-tuple integral, where the coefficients x^s denote that integration is to be performed x^1, x^2, $\cdots$ times over the oriented i-cells E_1^i, E_2^i, $\cdots$ respectively. However, we are not insisting on any specific interpretation of the symbol K^i.

A *chain* K of a complex Φ will be any linear combination of elementary i-chains of Φ, whether or not the elementary i-chains are all of the same dimensionality. Let us assign a distinctive symbol E_s to each elementary i-chain of Φ, irrespective of its dimensionality. A chain K will then be expressible by a relation of the form

$$K = x^s E_s, \tag{8.4}$$

where, according to our convention, summation with respect to the index s is to be understood.

A chain (8.4) will be said to *vanish*,

$$K=0,$$

if, and only if, its coefficients x^s are all zero. The *negative* of the chain (8.4) will be the chain $-x^sE_s$. The *sum* and *difference* of two chains x^sE_s and

$y^s E_s$ will be the chains $(x^s+y^s)E_s$, and $(x^s-y^s)E_s$ respectively. The chains

$$K_t=x_t^s E_s \qquad (t=1,3,\cdots,\tau)$$

will be said to be *linearly independent* if no linear combination of them with integer coefficients vanishes

$$\lambda^t K_t=0$$

unless all the coefficients λ^t are equal to zero.

9. **Boundaries of chains; homologies.** The *boundary* of an elementary i-chain $V_0V_1\cdots V_i$ will be defined as the $(i-1)$-chain

$$\sum_{s=1}^{i}(-1)^s V_0\cdots V_{s-1}V_{s+1}\cdots V_i\,. \tag{9.1}$$

Thus, if we use the notation

$$K\rightarrow K'^{*}$$

to indicate that K is bounded by K', we shall have

$$\begin{aligned} V_i &\rightarrow 0\,,\\ V_iV_j &\rightarrow V_j-V_i\,,\\ V_iV_jV_k &\rightarrow V_jV_k-V_iV_k+V_iV_j\,,\\ &\cdots\cdots\cdots\cdots\cdots\,. \end{aligned} \tag{9.2}$$

In the notation of § 8, the boundaries of the elementary i-chains E_s^i of a complex Φ are determined by relations of the form

$$E_s^i\rightarrow[i]_s^t E_t^{i-1} \tag{9.3}$$

where each of the coefficients $[i]_s^t$ is equal to 0, 1, or -1.

An i-chain K^i is, by definition, a linear combination of elementary i-chains E_s^i,

$$K^i=x^s E_s^i\,; \tag{9.4}$$

we define its *boundary* as the corresponding linear combination of the boundaries of the chains E_s^i:

$$K^i\rightarrow x^s[i]_s^t E_t^{i-1}\,. \tag{9.5}$$

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* Poincaré used the congruence symbol $\equiv$ in place of the arrow$\rightarrow$. The notation here adopted is perhaps less liable to cause confusion, and has the advantage of emphasizing the unsymmetrical character of the relation of bounding.

This definition is of the sort required for the correct formulation of the generalized Green-Stokes theorem which expresses certain i-tuple integrals over i-chains as $(i-1)$-tuple integrals over the boundaries of the i-chains.

A chain is always expressible as a sum of elementary i-chains, not necessarily of the same dimensionality. We define its *boundary* as the sum of the boundaries of its component elementary chains. It is clear that if

$$K_s \rightarrow K_s' \qquad (s=1, 2 \cdots, \sigma) \tag{9.6}$$

are bounding relations among chains, so also is any linear combination

$$\lambda^s K_s \rightarrow \lambda^s K_s'$$

of the relation (9.6). Moreover, if the coefficients μ^s and ν^s of a bounding relation

$$\mu^s K_s \rightarrow \nu^s L_s \tag{9.7}$$

have a common factor λ,

$$\mu^s = \lambda \bar{\mu}^s , \qquad \nu^s = \lambda \bar{\nu}^s ,$$

it is legitimate to divide both members of (9.7) by the factor λ and write

$$\bar{\mu}^s K_s \rightarrow \bar{\nu}^s L_s .$$

A chain will be said to be *closed* (as distinguished from *open*, or *bounded*) if its boundary vanishes. The boundary (9.1) of an elementary i-chain $V_0 V_1 \cdots V_i$ is always closed. For the boundary (9.1) of the i-chain is itself bounded by

$$\begin{aligned} &\sum_{t<s} \sum (-1)^{s+t} V_0 \cdots V_{t-1} V_{t+1} \cdots V_{s-1} V_{s+1} \cdots V_i \\ &+ \sum_{t>s} \sum (-1)^{s+t-1} V_0 \cdots V_{s-1} V_{s+1} \cdots V_{t-1} V_{t+1} \cdots V_i . \end{aligned} \tag{9.8}$$

But the two sums in (9.8) differ in sign, and in sign only, as we see by interchanging the names of the variables s and t in the second summation. More generally, the boundary of every chain is closed, since the boundary of a general chain K is the sum of the boundaries of the component elementary chains of K. According to our definition (cf. the first of relations (9.2)), every 0-chain is closed.* The boundary of an open i-chain is a closed $(i-1)$-chain.

* A different convention about the closure of 0-chains is also feasible, cf., for example, *VC*, p. 110. The convention adopted by Veblen amounts to replacing the relations $V_i \rightarrow 0$ in (9.2) by the symbolic relations $V \rightarrow 1$. The boundary of a general 0-chain $\mu^i V_i$ is then the sum of the coefficients μ^i of the chain, and the chain is closed if, and only if, this sum vanishes. Either convention has points in its favor, but the one adopted in this paper seems to be the more convenient one in the theory of intersecting chains.

A closed chain K of a complex Φ will be said to be *bounding*, or *homologous to zero*,

$$K \sim 0 \ , \ ^*$$

if it is the boundary of some open chain L of Φ. Two chains K and K', whether open or closed, will be said to be *homologous*,

$$K \sim K' \ ,$$

if their difference $K - K'$ is homologous to zero. Clearly, two chains with different boundaries can never be homologous.

Every linear combination of homologies

$$K_s \sim 0 \qquad (s = 1, 2, \cdots, \sigma)$$

is an homology:

$$\lambda^s K_s \sim 0 \ . \tag{9.9}$$

It is important to notice, however, that the homology $\lambda K \sim 0$ does not necessarily imply the homology $K \sim 0$, for a multiple λK of a chain K may bound even in cases where the chain K itself does not. In other words, *it is not generally permissible to divide through the coefficients of an homology by a common factor.*

A set of closed chains K_s will be said to be *linearly independent with respect to bounding* if no linear combination $\lambda^s K_s$ of them bounds, (9.9), unless all the coefficients λ^s of the combination vanish. Since a multiple λK of a chain K may bound although the chain K itself does not, it may happen that the set composed of a single non-bounding chain K is not linearly independent with respect to bounding.

10. **Connectivity numbers.** The maximum number of closed i-chains of a complex Φ that are linearly independent with respect to bounding will be called the *ith connectivity number* of the complex Φ and will be denoted by the symbol P^i.† We shall prove, presently, that the number P^i is a topological invariant of the complex Φ.

It is easy to see that the 0th connectivity number P^0 of a complex Φ is essentially positive and equal to the number of non-overlapping connected complexes out of which the complex Φ is formed. For, by the second relation in (9.2), no 0-chain $x^s E_s$ can bound unless the sum of all its coefficients is zero; consequently no elementary 0-chain can bound. Furthermore, two elementary 0-chains E_1^0 and E_2^0 belonging to the same connected portion

* We are here using the terminology and notation of Poincaré.

† Except for an additive constant equal to unity, this is simply the ith Betti number of Poincaré, *VC*, p. 110.

of Φ are always linearly dependent on one another, since the points to which they correspond may be connected by a broken line made up of vertices and edges, and since this broken line determines a 1-chain bounded by a 0-chain of the form $\pm E_1^0 \pm E_2^0$. Thus, a complete set of linearly independent 0-chains consists of one elementary 0-chain from each connected portion of Φ.

The *connectivity* P of a complex Φ will be defined as the sum with respect to i of all the connectivity numbers P^i of Φ. The number P is equal to the maximum number of linearly independent chains of the complex Φ.

It is a simple matter to calculate the connectivity P of a complex Φ from a knowledge of the fundamental relations

$$E_s \rightarrow \sum_t \mu_s^t E_t \qquad \begin{pmatrix} s,t=1,2,\cdots,\alpha \\ \mu_s^t=0,1,-1 \end{pmatrix} \tag{10.1}$$

determining the boundaries of the elementary chains of Φ. Let ρ be the rank of the matrix of coefficients μ_s^t and α the number of elementary chains E_s. Every closed chain is the left-hand member of a linear combination of relations (10.1) such that the right-hand member vanishes; hence the number of independent closed chains is $\alpha-\rho$. Moreover, every homology among the closed chains is a linear combination of the homologies

$$\sum_t \mu_s^t E_t \sim 0 \tag{10.2}$$

determined by the right-hand members of (10.1), since the boundary of an arbitrary chain K is equal to the sum of the boundaries of the elementary i-chains composing K. But the number of independent homologies (10.2) is equal to ρ. Therefore, finally, the maximum number of closed chains independent with respect to bounding is

$$P=\alpha-2\rho\ . \tag{10.3}$$

The individual connectivity numbers P^i may also be calculated in a similar manner. Let

$$\begin{aligned} E_s^{i+1} &\rightarrow \sum_t [i+1]_s^t E_t^i \\ E_t^i &\rightarrow \sum_u [i]_t^u E_u^{i-1} \end{aligned} \qquad \begin{pmatrix} s=1,2,\cdots,\alpha^{i+1} \\ t=1,2,\cdots,\alpha^i \\ u=1,2,\cdots,\alpha^{i-1} \end{pmatrix} \tag{10.4}$$

be the relations determining the boundaries of the elementary $(i+1)$- and i-chains of Φ respectively, and let ρ^{i+1} and ρ^i be the ranks of the matrices of coefficients $[i+1]_s^t$ and $[i]_t^u$ respectively. Then, by a similar argument to the last, we have

$$P^i=(\alpha^i-\rho^i)-\rho^{i+1}\ , \tag{10.5}$$

where, for $i=n$, we must put $\rho^{n+1}=0$. By eliminating the numbers ρ^i in relations (10.5) we obtain the so-called Euler-Poincaré formula,

$$\sum_{i=0}^{n}(-1)^i(P^i-\alpha^i)=0\ . \tag{10.6}$$

The above derivation of formulas (10.5) and (10.6) is due to Poincaré. The simplified appearance of (10.6) is due to the modified definition of the numbers P^i that we have adopted.

Since the boundary of each elementary $(i+1)$-chain is closed, we also have

$$E_s^{i+1}\to[i+1]_s^tE_t^i\to[i+1]_s^t[i]_t^uE_u^{i-1}=0\ ,$$

whence, the product of the matrices of coefficients $[i+1]_s^t$ and $[i]_t^u$ must vanish:

$$[i+1]_s^t[i]_t^u=0\ .* \tag{10.7}$$

For a similar reason, the square of the matrix of coefficients μ_s^t in (10.1) must also vanish,

$$\mu_s^t\mu_t^u=0\ . \tag{10.8}$$

11. **Coefficients of torsion.** The elementary i-chains E_s^t of a complex Φ are a *minimal base* of the set of all i-chains of Φ. If a set of i-chains F_s^t is also a minimal base, the chains F_s^t are expressible in terms of the chains E_s^t by relations of the form

$$F_s^i=\gamma_s^tE_t^i \qquad (\gamma_s^t \text{ integers})\ , \tag{11.1}$$

and the chains E_s^t in terms of the chains F_s^t by relations of the form

$$E_t^i=\bar{\gamma}_t^uF_u^i \qquad (\bar{\gamma}_t^u \text{ integers})\ . \tag{11.2}$$

Moreover, since the product of the transformations (11.1) and (11.2) is the identity, the matrix of coefficients γ_s^t is the inverse of the matrix of coefficients $\bar{\gamma}_t^u$; i.e.,

$$\gamma_s^t\bar{\gamma}_t^u=\delta_s^u \qquad \left(\delta_s^u=\begin{cases}0, & u\neq s\\ 1, & u=s\end{cases}\right).$$

It also follows from this that the product of the determinants $|\gamma_s^t|$ and $|\bar{\gamma}_t^u|$

* *VC*, p. 107.

is equal to unity. Therefore, the two determinants themselves are either both equal to $+1$ or both equal to -1.

For future reference, we recall the well known theorem that any transformation of the type (11.1) with determinant numerically equal to unity may be expressed as a product of *generating transformations* each of which is of one or the other of the following sorts:

(i) A transformation which leaves all but one of the members of a base invariant,

$$\overline{K}_s^i = K_s^i \qquad (s \neq j) ,$$

and transforms the remaining member K_j^i by the simple addition of another member of the base

$$\overline{K}_j^i = K_j^i + K_k^i \qquad (k \neq j) .$$

(ii) A transformation which leaves all but one of the members of a base invariant,

$$\overline{K}_s^i = K_s^i \qquad (s \neq j),$$

and merely changes the sign of the remaining member,

$$\overline{K}_j^i = -K_j^i .$$

Suppose we change the base of the i-chains of Φ according to (11.1) and the base of the $(i-1)$-chains according to

$$E_u^{i-1} = \bar{\epsilon}_u^v F_v^{i-1}, \qquad |\bar{\epsilon}_u^v| = \pm 1 . \tag{11.3}$$

Then we see from (11.1) and (11.3) that the fundamental relations of bounding,

$$E_t^i \rightarrow [i]_t^u E_u^{i-1} , \tag{11.4}$$

transform into

$$F_s^i \rightarrow (i)_s^v F_v^{i-1} , \tag{11.5}$$

where we have

$$(i)_s^v = \gamma_s^t [i]_t^u \bar{\epsilon}_u^v . \tag{11.6}$$

Now, the rank of the determinant of the coefficients $[i]_t^u$ in (11.4) is equal to ρ; therefore, by generating transformations on the i-chains E_s^i we may transform (11.4) into a set of new relations with the property that the right-hand members of all but the first ρ^i of these last relations are equal to zero. In other words, there exists a minimal base for the i-chains

consisting of ρ^i open chains L_t^i (such that every non-vanishing linear combination of the chains L_t^i is open) and $\alpha^i-\rho^i$ closed chains K_s^i. Let us choose a base of this sort for each value of i. We may then write the fundamental relations of bounding (11.4) in the form

$$(11.7)\qquad \begin{matrix} L_s^i \rightarrow (i)_s^t K_t^{i-1} \\ K_u^i \rightarrow 0 \end{matrix} \qquad \left(\begin{matrix} s=1,2,\cdots,\rho^i \\ t=1,2,\cdots,\alpha^{i-1}-\rho^{i-1} \\ u=1,2,\cdots,\alpha^i-\rho^i \end{matrix} \right).$$

We here use the fact, already noticed, that the boundary of an open i-chain is closed.

A further simplification of the first group of relations in (11.7) is possible. By generating transformation on the open i-chains L_s^i and on the closed $(i-1)$-chains K_t^{i-1}, we may reduce the matrix of coefficient $(i)_s^t$ to a normal form such that all elements are zero except the ones along the main diagonal, and such that these last elements are the elementary divisors of the matrix. The elements

$$(i)_1^1\ ,\ (i)_2^2\ ,\ \cdots$$

along the main diagonal of the normalized matrix have the property that each is divisible by its predecessors. The numerical values of the elementary divisors are called by Poincaré the *coefficients of torsion* of the complex Φ They are topological invariants of Φ. In our present treatment, we shall replace the connectivity numbers and coefficients of torsion by other invariants to be considered in the following section.

12. **Chains, modulo π.*** Now, let π be any integer greater than unity. By a *chain K, modulo π*, we shall mean any linear combination

$$(12.1)\qquad K=x^sE_s \qquad (\text{mod } \pi)\ ,$$

of elementary chains E_s with integer coefficients x^s reduced modulo π. The *boundary* of the chain K, modulo π, will be the boundary of the corresponding chain of the non-modular type, with coefficients reduced modulo π. A chain K, modulo π, will be *closed*, if its boundary vanishes,

$$(12.2)\qquad K\rightarrow 0 \qquad (\text{mod } \pi)\ ;$$

it will be *bounding*, or *homologous to zero*,

$$(12.3)\qquad K\sim 0 \qquad (\text{mod } \pi)\ ,$$

if it is the boundary of an open chain K', modulo π; and so on.

* Chains, modulo 2, were considered by Veblen and Alexander, *n-dimensional manifolds*, Annals of Mathematics, ser. 2, vol. 14 (1912–13), pp. 163–178. These chains have a particularly simple geometrical interpretation.

If the modulus π is a prime number, the discussion of § 10 obviously continues to hold when the term "chain" is reinterpreted to mean "chain, modulo π," for there is an almost perfect analogy between the theory of non-modular linear equations and the theory of linear equations to a prime modulus. Hence, when π is prime, we are led to certain new invariants $P^i(\pi)$ $(i=0, 1, \cdots, n)$, analogous to the connectivity numbers P^i. We shall call the number $P^i(\pi)$ the *ith connectivity number* of Φ, modulo π. It is equal to the maximum number of closed i-chains, modulo π, linearly independent with respect to bounding. The numbers $P^i(\pi)$ are given by formulas

$$P^i(\pi)=\alpha^i-\rho^i(\pi)-\rho^{i+1}(\pi)\ , \tag{12.4}$$

analogous to (10.5), where $\rho^i(\pi)$ and $\rho^{i+1}(\pi)$ are the ranks, modulo π, of the matrices of coefficients $[i]_s^t$ and $[i+1]_t^u$ respectively. Like the invariants P^i, the modular numbers $P^i(\pi)$ satisfy an Euler-Poincaré relation

$$\sum_{i=0}^{n}(-1)^i[P^i(\pi)-\alpha^i]=0 \tag{12.5}$$

analogous to (10.6). Their sum with respect to i gives the connectivity $P(\pi)$, modulo π, of the complex Φ.

As a matter of fact, it is fairly obvious that relations (12.4) and (12.5) may be derived even when the modulus π is not a prime, provided we put the proper interpretation on the rank $\rho(\pi)$, modulo π, of a matrix. It may be well, however, to say a few words about the theory of linear dependence to a general modulus π, since most treatments presuppose that π is a prime number.

A set of chains, modulo π,

$$K_1\ ,\ K_2\ ,\ \cdots\ ,\ K_\sigma \qquad (\mathrm{mod}\ \pi)\ , \tag{12.6}$$

will be *linearly independent* if no linear combination of them vanishes,

$$\sum^{\sigma}\lambda^s K_s=0 \qquad (\mathrm{mod}\ \pi)\ ,$$

unless all the coefficients λ^s vanish, modulo π. It should be noticed that, when π is not a prime, i. e.,

$$\pi=\lambda\tau\ , \tag{12.7}$$

a set consisting of a single non-vanishing chain K need not be linearly independent. For the chain K may be the τth multiple of a chain K', in which case we have by (12.7)

$$\lambda K=\lambda\tau K'=\pi K'=0 \qquad (\mathrm{mod}\ \pi).$$

By a *generating transformation* of a set of chains (12.6), modulo π, we shall mean a transformation of either of the following two sorts:

(i) A transformation which leaves all but one of the chains K_s invariant and alters the remaining one by the simple addition of another chain in the the set

$$\begin{aligned} \overline{K}_s &= K_s \\ \overline{K}_j &= K_j + K_k \end{aligned} \pmod{\pi} \tag{12.8}$$

$$(s = 1,2, \cdots, j-1, j+1, \cdots, \sigma \; ; j \neq k) \; .$$

(ii) A transformation which leaves all but one of the chains K_s invariant and multiplies the remaining one by an integer κ, such that the integer κ and modulus π are mutually prime,

$$\begin{aligned} \overline{K}_s &= K_s \\ \overline{K}_j &= \kappa K_j \end{aligned} \pmod{\pi} \tag{12.9}$$

$$(s = 1,2, \cdots, j-1, j+1, \cdots, \sigma \; ; j \neq k) \; .$$

Every generating transformation has an inverse which is expressible as a product of generating transformations. The inverse of (12.8) is

$$\begin{aligned} K_s &= \overline{K}_s \\ K_j &= \overline{K}_j - \overline{K}_s \end{aligned} \pmod{\pi}$$

$$(s = 1,2, \cdots, j-1, j+1, \cdots, \sigma \; ; j \neq k) \; ,$$

while the inverse of (12.9) is

$$\begin{aligned} K_s &= \overline{K}_s \\ K_j &= \lambda \overline{K}_j \end{aligned} \pmod{\pi} \; ,$$

where the coefficient λ is chosen to satisfy the diophantine equation

$$\lambda\kappa + \mu\pi = 1 \; .$$

Since the integers κ and π are mutually prime, this last equation determines the integer λ uniquely.

THEOREM. *Let*

$$K = \lambda^s K_s \pmod{\pi}$$

be a linear combination of chains K_s, modulo π, and

$$\overline{K} = \lambda^s \overline{K}_s = \mu^s K_s \pmod{\pi}$$

the transform of K under a generating transformation of the chains K_s. Then, if the coefficients λ^s do not all vanish, modulo π, neither do the coefficients μ^s.

The theorem follows at once from the definition of a generating transformation. It would cease to be true if the constant κ in a generating transformation of the second sort were allowed to have a factor in common with the modulus π.

COROLLARY. *A generating transformation carries a linearly independent set into a linearly independent set.*

For a relation $\lambda^s \overline{K}_s = 0 \pmod{\pi}$ would imply a relation $\mu^s K_s = 0 \pmod{\pi}$.

Two sets of chains K_s and L_s, modulo π, will be said to be *equivalent* if the set of all chains linearly dependent on the chains K_s is the same as the set of all chains linearly dependent on the chains L_s.

THEOREM. *A necessary and sufficient condition that two linearly independent sets of chains, modulo π, be equivalent is that either be transformable into the other by one or more generating transformations.*

The sufficiency of the condition is obvious. To prove its necessity, let K_s $(s=1, 2, \cdots, \sigma)$ and L_t $(t=1, 2, \cdots, \tau)$ be two linearly independent sets of chains, modulo π, and let the notation be so chosen that σ is at least as great as τ. Then, since the two sets are equivalent, each chain K_s is a linear combination of chains L_t,

$$K_s = \lambda_s^t L_t \pmod{\pi} . \tag{12.10}$$

Now, we know by the general theory of matrices that if the chains K_s and L_s were not reduced modulo π, it would be possible, by generating transformations on the K_s's and on the L_s's, to reduce relations (12.10) to the form

$$K'_s = \mu_s^t L'_t , \tag{12.11}$$

where all of the coefficients μ_s^t would vanish except the ones along the main diagonal of the matrix $||\mu_s^t||$. In the modular case, the corresponding generating transformations will therefore also reduce (12.10) to the form (12.11). In the modular case, we notice, further, that each of the coefficients μ_i^i along the main diagonal must be a prime residue, modulo π, for if we had

$$\mu_i^i q + \pi r = 0 \qquad \text{(not summed for } i) ,$$

we would have

$$qK'_i = \mu_i^i q L'_i = 0 \qquad \text{(not summed for } i) ,$$

whereas the chains K_i' are linearly independent, by the corollary to the last theorem. In particular, none of the coefficients μ_i^i can vanish; therefore, there must be exactly as many chains K_i' as there are chains L_i'. Finally, if all the coefficients μ_i^i along the main diagonal are prime residues, modulo π, the chains L_i' may be transformed into the chains K_i' by generating transformations of the second sort. It is sufficient, in fact, to multiply each chain L_i' by the inverse, modulo π, of the corresponding coefficient μ_i^i. Thus, the K_s's are equivalent to the K_s''s which are equivalent to the L_s''s, which are equivalent to the L_s's.

On the basis of these theorems, it follows immediately that when we are reducing to any modulus π, the fundamental relations of boundary

$$E_t^i \rightarrow [i]_t^u E_u^{i-1} \qquad (\text{mod } \pi)$$

may be reduced to a normal form

$$\begin{gathered} L_s^i \rightarrow (i)_s^t K_t^{i-1} \\ K^i \rightarrow 0 \end{gathered} \qquad (\text{mod } \pi)$$

$$\left(\begin{matrix} s = 1,2, \cdots, \rho^i(\pi) \\ t = 1,2, \cdots, \alpha^{i-1} - \rho^{i-1}(\pi) \\ u = 1,2, \cdots, \alpha^i - \rho^i(\pi) \end{matrix} \right), \tag{12.12}$$

analogous to (11.7), where all the coefficients $(i)_s^t$ vanish except those along the main diagonal, which last are not congruent to zero. The number of non-vanishing coefficients $(i)_s^s$ will be, by definition, the rank $\rho^i(\pi)$, modulo π, of the matrix $[i]_t^u$. If π is a prime, this definition is equivalent to the ordinary one.

If π is a prime, relations (12.12) may be reduced to such a form that all the diagonal elements $(i)_s^s$ are equal to unity. If π is not a prime, they may be reduced to such a form that each is exactly divisible by its predecessors (in the strict, non-modular sense) and that each is a factor of π. Thus, when π is not a prime, we obtain numbers analogous to the coefficients of torsion (§ 11).

The modular connectivity numbers $P^i(\pi)$ give exactly the same information about the complex Φ as the combined connectivity numbers P^i and coefficients of torsion. For, to derive the normalized modular relations of bounding (12.12) from the normalized non-modular ones (11.7), we must replace by a zero every coefficient of torsion in (11.7) containing the factor π, that is to say, every coefficient of torsion greater than or equal to π. Therefore, if $\sigma^i(\pi)$ is the number of coefficients of torsion of the ith order greater

than or equal to π, the rank of the matrix of coefficients $[i]_i^u$ is related to the rank, modulo π, of the same matrix in the following manner:

$$\rho^i = \rho^i(\pi) + \sigma^i(\pi) \ . \tag{12.13}$$

Consequently, by (10.5), (12.4), and (12.13), the modular connectivity number $P^i(\pi)$ is expressible in terms of the non-modular connectivity number P^i and coefficients of torsion according to the formula

$$P^i(\pi) = P^i + \sigma^i(\pi) + \sigma^{i+1}(\pi) \qquad [\sigma^0(\pi) = 0] \ . \tag{12.14}$$

On the other hand, the relations (12.14) may be solved successively for the numbers $\sigma^i(\pi)$ in terms of the connectivity numbers P^i and modular connectivity numbers $P^i(\pi)$. Moreover, for any particular complex Φ, there exists a positive integer π_0 such that

$$\rho^i(\pi) = \rho^i \qquad (\pi > \pi_0) \ ,$$

and hence,

$$P^i(\pi) = P^i \qquad (\pi > \pi_0 \ .) \tag{12.15}$$

Therefore, the connectivity numbers P^i and coefficients of torsion are functions of the modular connectivity number $P^i(\pi)$. Of course, it is never necessary to evaluate the invariants $P^i(\pi)$ for more than a finite number of values of π, in view of (12.15). While the theory of modular chains leads us to invariants $P^i(\pi)$ which are new in form only, we shall find it very serviceable in the developments to be treated in the second part of this paper.

III. Invariance of the topological constants

13. **Transformations of chains.** A *degenerate elementary i-chain* of a complex Φ will be defined as any symbol of the form

$$\pm V_0 V_1 \cdot \cdot \cdot V_i \tag{13.1}$$

containing at least one repeated vertex, where the vertices $V_0, V_1, \cdots, V_i$ are mutually adjacent vertices of Φ (§ 5). The *boundary* of a degenerate elementary i-chain will be defined according to the same law, (9.1), as the boundary of an ordinary elementary i-chain. A *generalized chain* will be any linear combination of elementary chains, whether ordinary or degenerate. Its boundary will be the sum of the boundaries of its component elementary chains.

Every elementary transformation τ (§ 5) carrying the vertices V_k of the complex Φ into vertices W_k of a complex Ψ may be thought of as carrying each elementary i-chain (13.1), whether ordinary or degenerate, into the elementary i-chain

$$\pm W_0 W_1 \cdot \cdot \cdot W_i \ . \tag{13.2}$$

It, therefore, carries every generalized chain of Φ into a generalized chain of Ψ. Moreover, if two generalized chains K and L of Φ are in the relation

$$K \rightarrow L \ , \tag{13.3}$$

they are carried respectively into chains K' and L' of Ψ such that

$$K' \rightarrow L' \ . \tag{13.4}$$

Now, we notice that the boundary of a degenerate elementary chain is a linear combination of degenerate elementary chains, just as the boundary of a non-degenerate elementary chain is a linear combination of non-degenerate elementary chains. For if a degenerate i-chain E^i with the boundary (9.1) contains more than two coincident vertices, every term in (9.1) contains coincident vertices, while if the chain E^i contains exactly two coincident vertices all but two of the terms in (9.1) contain coincident vertices, and the two which do not are of opposite signs, but otherwise identical, so that they cancel one another. Let us agree that the marks representing degenerate elementary chains are to be set equal to zero whenever they appear in the symbol for a generalized chain. Then, with this convention, a generalized chain reduces to an ordinary chain. Moreover, a relation of bounding, such as (13.3), is preserved under this reduction, for, by the remarks just made, the degenerate and non-degenerate portions of the left-hand member of (13.3) are bounded by the degenerate and non-degenerate portions of the right-hand member respectively. Thus, we may say that (13.3), regarded as a relation between ordinary chains, is carried by the transformation τ into (13.4), likewise regarded as a relation between ordinary chains. We conclude, therefore, that *an elementary transformation τ carries a closed chain into a closed (possibly vanishing) chain and a bounding chain into a bounding (possibly vanishing) chain.* Needless to say, however, the transformation τ may carry an open chain into a closed chain or a non-bounding chain into a bounding one.

It may be remarked, in passing, that the convention of setting degenerate elementary i-chains equal to zero is consistent with the notation of § 8, according to which an elementary chain $V_0V_1 \cdots V_i$ changes sign when any two of its vertices V_s are permuted.

14. **Invariance under elementary subdivisions.** Let ϵ be an elementary subdivision (§ 5) transforming a complex Φ into a complex Φ'. Then, as we recall, the subdivision ϵ is effected by introducing a new vertex W at the center of some i-cell

$$|V_0V_1 \cdots V_i| \tag{14.1}$$

of the complex Φ and subdividing each k-cell

$$|V_0V_1 \cdot \cdot \cdot V_iV_{i+1} \cdot \cdot \cdot V_k| \qquad (k \geqq i) \tag{14.2}$$

with the face (14.1) into the $i+1$ k-cells

$$|V_0 \cdot \cdot \cdot V_{t-1}WV_{t+1} \cdot \cdot \cdot V_iV_{i+1} \cdot \cdot \cdot V_k| \qquad (0 \leqq t \leqq i)\ , \tag{14.3}$$

together with certain cells of lower dimensionalities.

Now, let us agree to express each of the elementary chains of Φ corresponding to the cells (14.2) in terms of elementary chains of Φ' corresponding to the cells (14.3), in accordance with the following formulas:

$$\begin{aligned} &V_0 \cdot \cdot \cdot V_iV_{i+1} \cdot \cdot \cdot V_k \\ &= \sum_{t=0}^{i} V_0 \cdot \cdot \cdot V_{t-1}WV_{t+1} \cdot \cdot \cdot V_iV_{i+1} \cdot \cdot \cdot V_k \\ &= \sum_{t=0}^{i} (-1)^t WV_0 \cdot \cdot \cdot V_{t-1}V_{t+1} \cdot \cdot \cdot V_iV_{i+1} \cdot \cdot \cdot V_k\ . \end{aligned} \tag{14.4}$$

Since the remaining elementary chains of Φ are also elementary chains of Φ', it follows that the relations (14.4) enable us to express every chain K of Φ as a chain K' of Φ'.

LEMMA 1. *If the relations* (14.4) *identify a chain K of Φ with a chain K' of Φ', they also identify the boundary of the chain K with the boundary of the chain K'.*

To prove the lemma, it will be sufficient to show that the boundary of the left-hand member of each relation (14.4) may be identified with the boundary of the corresponding right-hand member (with the help of the other relations of the set); for a general chain K of Φ is merely a sum of elementary chains. The identification of the boundaries of the two members of (14.4) is self-evident geometrically. The formal proof is as follows.

The boundary of the left-hand member of (14.4) is

$$\sum_{t=0}^{k} (-1)^t V_0 \cdot \cdot \cdot V_{t-1}V_{t+1} \cdot \cdot \cdot V_k$$

which may be broken up into the two partial sums

$$\begin{aligned} &\sum_{t=0}^{i} (-1)^t V_0 \cdot \cdot \cdot V_{t-1}V_{t+1} \cdot \cdot \cdot V_k \\ &\qquad + \sum_{s=i+1}^{k} (-1)^s V_0 \cdot \cdot \cdot V_{s-1}V_{s+1} \cdot \cdot \cdot V_k\ . \end{aligned} \tag{14.5}$$

The boundary of the right-hand member is

$$
\begin{aligned}
&\sum_{t=0}^{i}(-1)^t V_0\cdot\cdot\cdot V_{t-1}V_{t+1}\cdot\cdot\cdot V_k\\
&+\sum_{s<t}\sum_{t=0}^{i}(-1)^{t+s+1}WV_0\cdot\cdot\cdot V_{s-1}V_{s+1}\cdot\cdot\cdot V_{t-1}V_{t+1}\cdot\cdot\cdot V_k\\
&+\sum_{t<s}\sum_{s=0}^{i}(-1)^{t+s}WV_0\cdot\cdot\cdot V_{t-1}V_{t+1}\cdot\cdot\cdot V_{s-1}V_{s+1}\cdot\cdot\cdot V_k\\
&+\sum_{t=0}^{i}\sum_{s=i+1}^{k}(-1)^{t+s}WV_0\cdot\cdot\cdot V_{t-1}V_{t+1}\cdot\cdot\cdot V_{s-1}V_{s+1}\cdot\cdot\cdot V_k\,.
\end{aligned}
\tag{14.6}
$$

But the second and third sums in (14.6) cancel one another, as may be seen by interchanging the names of the variables s and t in the second. Moreover, the first sum in (14.6) is identical with the first in (14.5) and the fourth sum in (14.6) with the second in (14.5) when this last sum is expressed as a chain of Φ', by means of relations (14.4). Thus, we have identified the boundaries of the left and right-hand members of each relation (14.4), and thereby proved the lemma.

In the notation that we have just been using, let $|V_0\cdot\cdot\cdot V_k|$ be a k-cell of the complex Φ such that the new vertex W lies at the center of an i-face $|V_0\cdot\cdot\cdot V_i|$ of $|V_0\cdot\cdot\cdot V_k|$. Properly speaking, there exists no elementary $(k+1)$-chain

$$WV_0\cdot\cdot\cdot V_k\,, \tag{14.7}$$

since the points $W, V_0, \cdot\cdot\cdot, V_k$ cannot be the vertices of a $(k+1)$-cell. For symmetry of expression, however, we shall speak of (14.7) as a *symbolic* $(k+1)$-chain. The *boundary* of the symbolic chain (14.7), formed according to (9.1), is

$$
\begin{aligned}
V_0\cdot\cdot\cdot V_k&-\sum_{t=0}^{i}(-1)^t WV_0\cdot\cdot\cdot V_{t-1}V_{t+1}\cdot\cdot\cdot V_k\\
&-\sum_{t=i+1}^{k}(-1)^t WV_0\cdot\cdot\cdot V_{t-1}V_{t+1}\cdot\cdot\cdot V_k\,,
\end{aligned}
$$

where the first two terms are made up of ordinary k-chains, while the last is made up of symbolic ones. Moreover, by (14.4), the first two terms cancel one another, so that the boundary of the symbolic $(k+1)$-chain (14.7) reduces to a sum of symbolic k-chains,

$$\sum_{t=i+1}^{k}(-1)^t WV_0\cdot\cdot\cdot V_{t-1}V_{t+1}\cdot\cdot\cdot V_k\,. \tag{14.8}$$

In view of this fact, we may operate with symbolic chains in just the same way as with degenerate chains, by regarding them merely as symbols for zero.

A chain K will be said to *meet* a vertex V if the symbol $\pm V_0V_1\cdots V_i$ of one of its component elementary chains contains the mark V.

LEMMA 2. *Every chain K' of Φ' such that its boundary does not meet the new vertex W is homologous with a chain K of Φ.*

This proposition is also self-evident geometrically. The detailed proof is as follows:

We break up the chain K' into a pair of chains J' and L,

$$K'=J'+L\,, \tag{14.9}$$

composed respectively of the elementary chains of K' that meet the new vertex and of those that do not. The chain J' is of the form

$$J'=\sum\pm WV_{s_1}\cdots V_{s_q}\,; \tag{14.10}$$

the boundary of J' is of the form

$$\begin{aligned} B=&\sum\pm V_{s_1}\cdots V_{s_q}\\ &-\sum_{t=1}^{q}\ \sum\pm(-1)^t WV_{s_1}\cdots V_{s_{t-1}}V_{s_{t+1}}\cdots V_{s_q}\,. \end{aligned} \tag{14.11}$$

Moreover, since the boundaries of K' and L do not meet the vertex W, neither does the boundary B of J'. Therefore, the double sum in (14.11) must vanish by mutual cancellation of its terms,

$$\sum_{t=1}^{q}\ \sum\pm(-1)^t WV_{s_1}\cdots V_{s_{t-1}}V_{s_{t+1}}\cdots V_{s_q}=0 \tag{14.12}$$

leaving the relation

$$B=\sum\pm V_{s_1}\cdots V_{s_q}\,. \tag{14.13}$$

Now, let V_0 be any vertex of the cell of the complex Φ on which the new vertex W lies. Consider the sum of elementary chains

$$\sum\pm V_0WV_{s_1}\cdots V_{s_q} \tag{14.14}$$

(actual, degenerate, or symbolical) obtained by prefixing the mark V_0 to each elementary chain in J', (14.10). The boundary of (14.14) is

$$\begin{aligned} &\sum\pm WV_{s_1}\cdots V_{s_q}-\sum\pm V_0V_{s_1}\cdots V_{s_q}\\ &-\sum\ \sum\pm(-1)^t V_0WV_{s_1}\cdots V_{s_{t-1}}V_{s_{t+1}}V_{s_q}\sim 0\,. \end{aligned} \tag{14.15}$$

But the double sum in (14.15) must vanish by mutual cancellation of its terms, since it differs from the double sum in (14.12) merely by the presence of the mark V_0 at the head of each term. Therefore, (14.15) reduces to

$$J'-M\sim 0 \qquad (M=\sum \pm V_0 V_{s_1}\cdot\cdot\cdot V_{s_q})\ . \tag{14.16}$$

From (14.9) and (14.16), we finally obtain

$$K'\sim M+L\ .$$

But the chains M and L do not meet the vertex W. Therefore, $M+L$ is a chain of the complex Φ, which proves the lemma.

15. **Invariance of the topological constant.** Let Φ and Ψ be two homeomorphic complexes. Then, by the corollary at the end of § 7, there exist derived complexes Ψ_j and Φ_k of Ψ and Φ respectively and elementary transformations τ and τ' such that (1) the transformation τ carries vertices of Ψ_j into vertices of Φ, (2) the transformation τ' carries vertices of Φ_k into vertices of Ψ_j, (3) the transformation $\tau'\tau$ is pseudo-identical.

LEMMA. *Every i-chain K of the complex Φ, when expressed as a chain of Φ_k and operated on by the pseudo-identical transformation $\tau'\tau$, is left invariant by the transformation $\tau'\tau$.*

The proof is made by induction with respect to i. If i is zero, the lemma is immediate, since the pseudo-identical transformation $\tau'\tau$ leaves invariant the vertices of Φ. Let us, therefore, assume that the lemma has been proved for all k-chains of dimensionalities k less than i and show, first of all, that it is true for an elementary i-chain E^i corresponding to an i-cell C^i of Φ. We shall denote the boundary of E^i by K^{i-1},

$$E^i\rightarrow K^{i-1}\ .$$

Now, the chain E^i, when expressed as a chain of Φ_k, consists of a sum of elementary i-chains corresponding to sub-cells of C^i. Moreover, each vertex of each of these sub-cells is carried by the transformation $\tau'\tau$ into a vertex of C^i; therefore, E^i must be carried into some multiple λE^i of itself, such that

$$\lambda E^i\rightarrow\lambda K^{i-1}\ .$$

However, by the hypothesis of the induction, the boundary K^{i-1} of E^i is carried into itself. Therefore, $\lambda=1$. Therefore, finally, E^i is carried into itself. It follows at once that every chain K of Φ is left invariant by the transformation $\tau'\tau$ since each of the component elementary chains of K is left invariant.

We may now complete the proof of the invariance of the connectivity numbers in a few lines. Let

$$K_s^i \qquad (s=1, 2, \cdots, P^i)$$

be a complete set of closed i-chains of the complex Φ, linearly independent with respect to bounding. These chains, expressed as chains of Φ_k, are carried by the transformation τ' into chains

$$L_s^i, \qquad (s=1, 2, \cdots, P^i)$$

of Ψ_j respectively. Moreover, the chains L_s^i are carried back again into the chains K_s^i respectively by the transformation τ, in consequence of the lemma. It follows that the chains L_s^i must be linearly independent with respect to bounding; for a relation of bounding

$$\lambda^s L_s^i \sim 0$$

between the chains L_s^i would go over into a relation of bounding

$$\lambda^s K_s^i \sim 0$$

between the chains of K_s^i, contrary to hypothesis. Thus the connectivity number P^i must be at least as great for the complex Ψ as for the complex Φ. But, by reversing the argument, the number P^i must be at least as great for the complex Φ as for the complex Ψ. Hence, finally, the number P^i must be the same for the two complexes.

A similar argument proves the invariance of the modular connectivity numbers $P^i(\pi)$ and, hence, of the coefficients of torsion.

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